# Simple algorithm to test for linking to Wilson loops in percolation

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A simple burning or epidemic type of algorithm is developed in order to test whether any loops in percolation clusters link a fixed reference loop, a problem considered recently by Gliozzi *et al.* in the context of gauge theory. We test our algorithm at criticality in both two dimensions, where the behavior agrees with a theoretical prediction, and in three dimensions.

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## I. INTRODUCTION

Recently, Gliozzi *et al.* [1,2] have studied percolation in the context of gauge theory. They considered the question of whether closed paths in three-dimensional (3D) percolation clusters are linked topologically to given closed loops, the so-called Wilson loops. Studying this problem for rectangular planar loops, and in comparison to percolation in three-dimensional slabs which they relate to the problem of deconfinement, the authors find a universal amplitude ratio. This work provides an example where the percolation model possesses connections to fundamental problems in theoretical particle physics.

Gliozzi *et al.*'s numerical results for rectangular loops of dimensions $R\times T$ confirmed the expected behavior for $p \neq p_c$ [3],

$$\langle W(R,T)\rangle = Ce^{-P(R+T)-\sigma RT}R^{1/4}\sqrt{\frac{\eta(i)}{\eta(iT/R)}}, \tag{1}$$

where $\langle W(R,T)\rangle$ is the average probability that there is no path in any cluster linked to the Wilson loop; $C$, $P$, and $\sigma$ are constants that depend upon the percolation probability $p$; and $\eta$ is the Dedekind function $\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ with $q=e^{2i\pi\tau}$. When $p<p_c$, one expects $\sigma=0$ because the linking probability should depend only upon the perimeter of the loop, while for $p>p_c$, the linking probability is expected to decay exponentially with the area of the loop ($\sigma>0$). Taking $p$ somewhat above $p_c$, Gliozzi *et al.* found that the dependence of $\sigma$ upon $p$ behaves as

$$\sigma = S(p-p_c)^{2\nu}, \tag{2}$$

similar to the behavior of a surface tension, where $\nu \approx 0.876\,5$ is the correlation-length exponent of 3D percolation [4], and $S$ is a constant. They also determined the percolation threshold $p_\ell$ for slabs of thickness $\ell$ in the range 3–8. One expects $\ell \sim \xi(p_\ell) \sim T_c^{-1}(p_\ell - p_c)^{-\nu}$, where $T_c$ is a constant, and indeed they find $1/(\ell\sqrt{\sigma(p_\ell)}) \sim T_c/\sqrt{S}$ is a universal amplitude ratio with a value of about 1.50.

In this paper, we discuss two points related to the work of Gliozzi *et al.*: (1) We describe an epidemic or burning [5] type of algorithm that may be simpler than the algorithm described by Gliozzi *et al.*, and (2) we apply it to study the linking probability exactly at $p_c$ (a point that Gliozzi *et al.* did not consider) for two-dimensional (2D) and 3D systems. Note that Eq. (1) and its 2D analog are not necessarily expected to be valid at $p_c$.

## II. ALGORITHM

Gliozzi *et al.* describe an algorithm that involves successive removal of dangling ends and reduction to an auxiliary graph that represents the connections between clusters on either side of the flat region enclosed by the loop. This graph is used to determine whether a cluster is linked to the loop.

Here we describe a cluster burning type of algorithm that accomplishes the same test. As in Ref. [1], we consider the loop $\gamma$ to be on the dual lattice, so effectively the problem is to find if there are clusters that simultaneously pass through the plane $\Sigma$ of vertical bonds enclosed by $\gamma$ and through bonds in the same plane outside of $\gamma$. We are thinking of a simple cubic lattice to be specific.

To begin the process, all of the sites are set to the "unvisited" state and bonds to the "undetermined" state. Then we pick one of the (unvisited) sites $\mathcal{S}$ directly above $\Sigma$, and label that site as "visited" with an arbitrary index $n$. We check the six bonds that emanate from $\mathcal{S}$; the undetermined bonds are made "occupied" with probability $p$ and "vacant" otherwise. For the bonds that are occupied, we check the adjacent site; if that site is unvisited, we label it as visited (with a value of the label described below) and put its coordinates on a queue for future checking. After finishing checking all bonds connected to the site being studied, we consider the next site on the queue, continuing this process until the queue is empty. This is the normal burning or epidemic type of algorithm to identify a cluster connected to a site in bond percolation; here we also decide whether a bond is occupied or not as we go along. We repeat this process for all remaining unvisited sites in the plane above $\Sigma$.

What we now do differently for the loop-linking problem is that we assign an index $n$ to each visited site in a cluster. When we transverse one of the occupied bonds that intersects $\Sigma$, we increment $n$ by one when going downward or decrement it by one when going upwards. In this way, every site of the cluster will be labeled by $n, n\pm 1$ (if a path of the growing cluster goes once through $\Sigma$), $n\pm 2$ (if a path of the growing cluster winds twice through $\Sigma$), etc.

Now, if during the growing process a new bond is found to connect two visited sites with different labels $\mathcal{L}$, then the cluster must have wrapped around $\gamma$ and is therefore linked to it.

An occupied bond of course will not connect to sites of two different clusters, by definition, so therefore one does not have to worry about interference between clusters in this algorithm.

Here we have not dealt with the system boundaries. If open boundaries are used, care must be taken so that the boundary bonds are not mistaken for wraparounds. To eliminate this problem and to lessen the effects of the boundary, we considered periodic boundary conditions in all directions. This makes the problem slightly different, because periodic wraparounds through the Wilson loop will also contribute to linking events; but if the lattice dimension $L \gg R$, this difference should not be too significant. Indeed, a moment's reflection shows that is very unlikely that there will be wraparound without linking, since a cluster that wraps around the lattice it is usually a ubiquitous one and most likely will link the Wilson loop also.

The idea of adding a label to sites in percolation to test for a crossing criterion has been used previously in relation to wrapping a periodic system in a given direction [6,7].

### III. LINKING IN TWO DIMENSIONS AT CRITICALITY

For the 2D system, the question that is studied is whether there exists a closed path in a cluster that encircles one (but not both) of two points on the dual lattice, separated by a distance $R$. In this case we can make a simple theoretical prediction for $\langle W(R)\rangle$, since the condition of a path not encircling either of the two points individually is equivalent to the existence of a continuous path between the two points on the dual lattice—that is, a cluster that connects the two points. The density drop off from any point on a given cluster goes as $r^{D-d}$, where $D$ is the fractal dimension and $d$ is the spatial (Euclidean) dimension. To find the probability that two given points are connected, the above factor must be multiplied by the probability that the size of a cluster connected to one of the points is at least large enough to reach the other point. At criticality, the probability that the number of sites connected to a point is equal to or greater than $s$ is given by $P_{\geq s} \sim s^{2-\tau}$, where $\tau$ is the size distribution exponent, and this implies that the probability that the radius is greater than or equal to $r$ is given by $P_{\geq r} \sim r^{D(2-\tau)}$, since $s \sim r^D$. Then, by the hyperscaling relation $d/D = \tau - 1$, we have $P_{\geq r} \sim r^{D-d}$. Thus, the net probability that two points separated by $r$ are connected by a cluster at $p_c$ is given by $P(r) \sim r^{2(D-d)}$, implying that

$$\langle W(R)\rangle \sim R^{-2(d-D)} = \exp[-2(d-D)\ln R]. \qquad (3)$$

In $d=2, D=91/48$ and $2(d-D) \approx 0.208$.

We carried out simulations for this system using the algorithm described above. We considered bond percolation on the square lattice at $p=p_c=1/2$, on a system with a square boundary of dimensions $1024 \times 1024$, and considered separations of the two points ranging between 10 and 100. Figure 1 shows the results for a plot of ln $W$ vs ln $R$, for a relatively small number of runs (100 000 each). The slope is about $-0.22$, consistent with the theoretical prediction above. To make this work more precise, one would have to consider different size systems to study the finite-size corrections, and perhaps also consider systems with open boundary conditions for comparison.

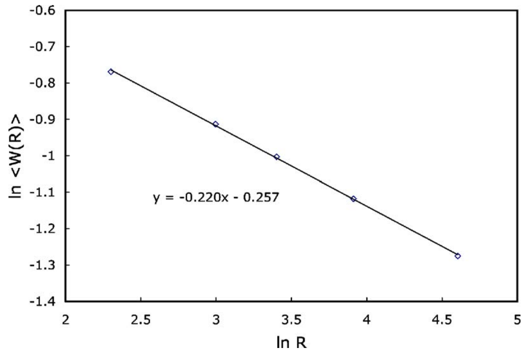


FIG. 1. (Color online) Logarithm of $\langle W(R)\rangle$ (=the probability that no cluster encircles just one of the two points) vs the logarithm of the points' separation $R$.

### IV. LINKING IN THREE DIMENSIONS AT CRITICALITY

For the 3D problem, we consider bond percolation on the simple cubic lattice, and take $p=0.248\,8126$, which is an estimate for $p_c$ believed to be within about $5\times10^{-7}$ of the actual value [8]. We consider a lattice of size $128\times128\times128$, and square Wilson loops containing $R\times R$ vertical bonds, with $R=5$, 11, 21,…, 111. Between 300 000 (smaller $R$) and 13 000 000 samples (larger $R$) were generated for the different values of $R$.

In Fig. 2, the lower curve represents $\ln\langle W(R,R)\rangle$ as a function of $R$. The data shows quite linear behavior up to $R=111$. Evidently, the perimeter term proportional to $P$ in Eq. (1) dominates; as expected, there is no term proportional to the area.

We do not see evidence of the $R^{1/4}$ term in Eq. (1). The data marked by circles in Fig. 2 represents $\ln(R^{-1/4}\langle W(R,R)\rangle)$ vs $R$, and the fit to a straight line is much worse than for the case without the factor of $R^{-1/4}$. This factor would show up as a logarithmic term in the plot of Fig. 2.

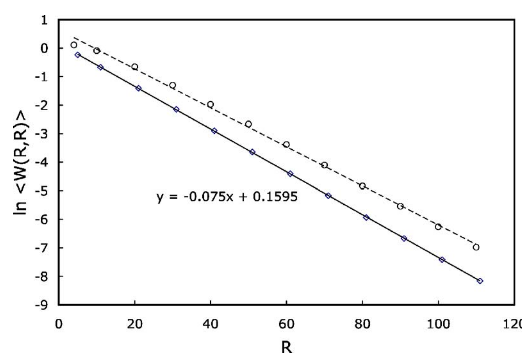


FIG. 2. (Color online) 3D data. Upper (○): $\ln(R^{-1/4}\langle W(R,R)\rangle)$ vs $R$. Lower (◇): $\ln\langle W(R,R)\rangle$ vs $R$, which shows a good fit to a straight line for $5\leq R\leq 111$. The equation of the linear fit is given.

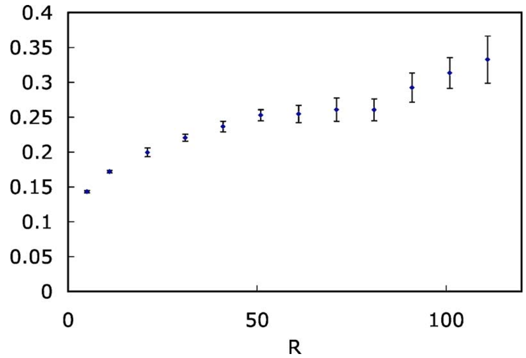


FIG. 3. (Color online) $\ln W + 0.0765\,R$ vs $R$ for the 3D data, showing deviations from simple exponential behavior. Error bars show two standard deviations of statistical error.

To check further for logarithmic terms, we plot in Fig. 3 the quantity $\ln W + 0.0765\,R$, where the constant 0.0765 was adjusted to get the best horizontal region in the center, along with general monotonic behavior. We see two corrections to the straight line: for small $R$, there is a small decrease, which could be fit to a very small logarithmic term, $\approx -0.03 \ln R$, much smaller than the $-(1/4)\ln R$ term that would appear for $p > p_c$ according to Eq. (1). The coefficient is so small that the existence of a logarithmic term seems unlikely.

For large $R$, the deviations from linearity are also small, which is surprising given that we went up to $R=111$ in a system of size $L=128$. When $R$ approaches $L$ it should be more difficult to create a linking cluster, because there is a smaller region external to the loop (also taking into account the periodic boundary conditions), but this may be balanced by the increase in vertical wraparounds (which our algorithm takes to be linkages) through the periodic boundary conditions. Note that at $R=111, W=0.000\,286\,17$—that is, only 3434 of the 12 000 000 samples did not have a linkage to the Wilson loop.

Thus, ignoring the possible small logarithmic term, the data for three dimension (in the central range) yields $\ln W = -0.0765\,R + 0.25$, implying

$$\langle W(R,R)\rangle = 1.28 e^{-0.0382(2R)} \tag{4}$$

or $P=0.0382$ for bond percolation on the simple cubic lattice at criticality. Note the linear fit for $W$ above is somewhat different than that given in Fig. 2, which is just a simple linear fit through all the data points.

## V. CONCLUSIONS

We see that a simple burning type of algorithm can be constructed to find the loop-linking probability studied by Gliozzi *et al.* We have checked it in two dimensions at the critical threshold, where the linking probability is known exactly by virtue of its being dual to the two-point probability. Of course, in two dimensions one can easily simulate the dual problem of connecting the two points. However, in three dimensions, where a dual-lattice procedure would be much more complicated, a direct determination is preferable and the algorithm presented here is efficient and simple. For three dimensions, we find a simple exponential relation between $\langle W(R,R)\rangle$ and $R$ reflecting a perimeter effect; there is no evidence of a logarithmic correction implied by the $R^{1/4}$ term in Eq. (1) (which is not necessarily expected to be valid at $p_c$) or as suggested by the behavior in two dimensions.

## ACKNOWLEDGMENTS

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